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Design of a trilaminated rectangular fin

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Abstract—A continuum mixture theory combined with the linear operator method is used to solve the problem of transient heat conduction in a rectangular trilaminated fin. Applying the continuum mixture theory and the linear operator method reduces the problem to that of a coupled Sturm–Liouville problem. A method is developed to find eigenvalues and eigenfunctions of the coupled Sturm–Liouville problem. Transient conduction in both symmetrical and asymmetrical rectangular fins is illustrated. It turns out that the asymmetrical fin performs better than the symmetrical one in terms of efficiency.

INTRODUCTION

A FIN has been widely used as a device to increase the rate of heat transfer between an object and its surrounding fluid. The problem of transient heat conduction in a fin was solved a long time ago by various investigators [1–5]. The one-dimensional approximation to heat conduction in a fin is valid only when the Biot number is small [6, 7]. Chu *et al.* [8] studied the transient conduction in a rectangular fin by using the Laplace transformation method and obtained Laplace inversion by the numerical method. The validity of numerical inversion of the Laplace transformation was discussed by Davis and Martin [9]. Recently Ju *et al.* [10] applied continuum mixture theory [11–13] to solve the transient conduction in a rectangular fin and found that the method was very efficient in reducing the two-dimensional problem of transient heat conduction in a rectangular fin to that of a one-dimensional problem.

Under certain circumstances, the fin is in contact with high temperature or corrosive fluid. A layer of high strength or corrosion resistant material is coated on each side of the fin to withstand the harsh environment. Barker [14] studied the steady-state heat conduction in a rectangular composite fin. Chu *et al.* [15] solved the problem of transient conduction in a trilaminated rectangular fin by using the Laplace transformation and separation of variables methods. The accuracy of their solution is dubious since the numerical inversion of the Laplace transformation is tedious and the series converges very slowly [16].

In this work, a continuum mixture theory combined with a linear operator method were applied to solve the transient conduction in a trilaminated rectangular fin. A method is developed to solve the resulting coupled Sturm–Liouville problem. Though there is a maximum of 30% error between results of this work and those presented by Chu *et al.* [15], close agreement between the results of this work and the numerical solution of Ghoshdastidar and Mukhopadhyay [17] confirms the correctness of the results of this work.

PROBLEM FORMATION AND SOLUTION

Consider a rectangular fin composed of three layers of different material as shown schematically in Fig. 1. The following assumptions are applied.

(1) The convective heat transfer coefficients on two sides of the fin h_0 and h_3 are constant.

(2) The rate of heat transfer from the tip of the fin is negligible.

(3) All physical properties are constant.

(4) Perfect contact at the interface between layers of different material.

(5) For $t \leqslant 0$, the fin is in thermal equilibrium with the surrounding fluid. For $t > 0$, the base of the fin is subject to a step change in temperature.

The equation of energy conservation for the fin can be expressed as

$$\rho_i^* c_{pi}^* \frac{\partial T_i^*}{\partial t^*} + \frac{\partial q_{x_i}^*}{\partial x^*} + \frac{\partial q_{y_i}^*}{\partial y^*} = 0, \quad i = 1, 2, 3 \qquad (1)$$

where the constitutive equations are

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NOMENCLATURE

a_i	constant, equation (25)
Bi	Biot number, $h_0/\varepsilon = h_3/\varepsilon$
b_i	constant, equation (26)
b^*	thickness of fin
$\underline{\underline{C}}$	matrix, equation (43)
c_i	constant, equation (28)
c_j^i	first component of eigenvector W_j^i
c_p^*	heat capacity of fin
$D(L)$	domain of L
$\underline{\underline{D}}$	matrix, equation (43)
d_j^i	second component of eigenvector W_j^i
$\underline{\underline{E}}$	matrix, equation (43)
e_j^i	third component of eigenvector W_j^i
$\underline{f}$	vector, equation (44)
$\underline{G}$	vector, equation (44)
H_i	subspace
h_0	convection heat transfer coefficient
h_3	convection heat transfer coefficient
$\mathbf{H}$	Hilbert space
k^*	thermal conductivity of fin
K	thermal conductivity ratio of base material to coating material
l^*	length of fin
L	operator, equation (44)
$\mathbf{L}$	self-adjoint operator, $\{L, D(L)\}$
P_i	weighting function
P_{ij}	ijth component of the matrix $\underline{\underline{P}}$
P_{ij}^{-1}	ijth component of matrix $\underline{\underline{P}}^{-1}$
q_x	heat flux in the x-direction
q_y	heat flux in the y-direction
Q	heat transfer rate
T	temperature of fin
T_∞	temperature of ambient fluid
T_0	initial temperature of fin
t	time
x	x coordinate
y	y coordinate.

Greek symbols

α	thermal diffusivity ratio of base material to coating material
ε	perturbation parameter, b^*/l^*
θ	temperature defined in equation (44)
λ	eigenvalue
ρ^*	density of fin.

Subscripts

i	ith layer of the trilaminated fin
j	jth eigenvalue or eigenvector
$-$	vector
$=$	matrix.

Superscripts

$\bar{}$	average, equation (31)
i	ith branch of eigenvalue
T	transpose of matrix
$'$	differentiation
$*$	dimensional quantity.

$$\left.\begin{aligned} q_{x_i}^* &= -k_i^* \frac{\partial T_i^*}{\partial x^*} \qquad (2)\\ q_{y_i}^* &= -k_i^* \frac{\partial T_i^*}{\partial y^*} \qquad (3)\end{aligned}\right\} \quad i = 1,2,3$$

with boundary conditions

$$y^* = 0, \quad q_y^* = h_0^*(T_\infty^* - T_0^*) \qquad (4)$$

$$\left.\begin{aligned} y^* = y_i^*, \quad q_{y_i}^* &= q_{y_{i+1}}^* \qquad (5)\\ y^* = y_i^*, \quad T_i^* &= T_{i+1}^* \qquad (6)\end{aligned}\right\} \quad i = 1,2$$

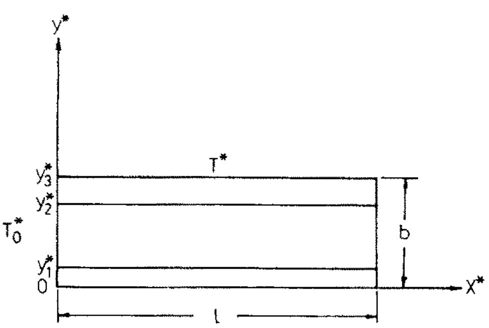

FIG. 1. Schematic diagram of trilaminated fin.

$$y^* = y_3^*, \quad q_y = h_3^*(T_3^* - T_\infty^*) \qquad (7)$$

$$\left.\begin{aligned} x^* &= 0, \quad T_i^* = T_0^* \qquad (8)\\ x^* &= l^*, \quad \frac{\partial T_i^*}{\partial x^*} = 0 \qquad (9)\end{aligned}\right\} \quad i = 1,2,3$$

and initial conditions

$$t^* = 0, \quad T_i^* = T_\infty^*, \quad i = 1,2,3. \qquad (10)$$

Equations (5) and (6) are interface conditions which state that flux and temperature are continuous at the interfaces.

By defining the following dimensionless variables:

$$\varepsilon = \frac{b^*}{l^*}, \quad x = \frac{x^*}{l^*}, \quad y = \frac{y^*}{b^*}, \quad T_i = \frac{T_i^*}{T_{\mathrm{ref}}^*}, \quad k_i = \frac{k_i^*}{k_{\mathrm{ref}}^*}$$

$$\rho_i c_{pi} = \frac{\rho_i^* c_{pi}^*}{(\rho^* c_p^*)_{\mathrm{ref}}}, \quad q_{\mathrm{ref}}^* = \frac{k_{\mathrm{ref}}^* T_{\mathrm{ref}}^*}{l^*}, \quad t = \frac{k^* t^*}{(\rho^* c_p^*)_{\mathrm{ref}} l^2}$$

$$q_{x_i} = \frac{q_{x_i}^*}{q_{\mathrm{ref}}^*}, \quad q_{y_i} = \frac{q_{y_i}^*}{\varepsilon q_{\mathrm{ref}}^*}.$$

Equations (1)–(10) become

$$\rho_i c_{pi} \frac{\partial T_i}{\partial t} + \frac{\partial q_{x_i}}{\partial x} + \frac{\partial q_{y_i}}{\partial y} = 0, \quad i = 1,2,3 \qquad (11)$$

$$\left.\begin{aligned} q_{x_i} &= -k_i\frac{\partial T_i}{\partial x} && (12)\\ \varepsilon^2 q_{y_i} &= -k_i\frac{\partial T_i}{\partial y} && (13)\end{aligned}\right\}\quad i=1,2,3$$

$$y=0,\quad q_y=\frac{h_0}{\varepsilon^2}(T_\infty - T_1) \tag{14}$$

$$\left.\begin{aligned} y=y_i,\quad q_{y_i} &= q_{y_{i+1}} && (15)\\ y=y_i,\quad T_i &= T_{i+1} && (16)\end{aligned}\right\}\quad i=1,2$$

$$y=y_3,\quad q_y=\frac{h_3}{\varepsilon^2}(T_3 - T_\infty) \tag{17}$$

$$\left.\begin{aligned} x=0,\quad & T_i = T_0 && (18)\\ x=1,\quad & \frac{\partial T_i}{\partial x}=0 && (19)\\ t=0,\quad & T_i=T_\infty && (20)\end{aligned}\right\}\quad i=1,2,3.$$

Since $\varepsilon = b^*/l^* \ll 1$, we use the regular perturbation method and express each dependent variable in the following form:

$$f=\sum_{n=1}^{\infty}\varepsilon^{2n}f^{(n)}. \tag{21}$$

After expressing each dependent variable in the form as shown above, we have the zeroth (ε^0) order equations for equations (11)–(13) as

$$\rho_i c_{pi}\frac{\partial T_i^{(0)}}{\partial t}+\frac{\partial q_{x_i}^{(0)}}{\partial x}+\frac{\partial q_{y_i}^{(0)}}{\partial y}=0 \tag{22}$$

$$k_i\frac{\partial T_i^{(0)}}{\partial y}=0 \tag{23}$$

$$q_{x_i}^{(0)} = -k_i\frac{\partial T_i^{(0)}}{\partial x}. \tag{24}$$

From equations (23) and (24), we have $T_i^{(0)} = T_i^{(0)}(x,t)$, $q_{x_i}^{(0)} = q_{x_i}^{(0)}(x,t)$, then from equation (22) we have

$$\frac{\partial q_{y_i}^{(0)}}{\partial y}=a_i \tag{25}$$

and hence

$$q_{y_i}^{(0)} = a_i y + b_i \tag{26}$$

where $a_i = a_i(x,t)$, $b_i = b_i(x,t)$. From the first-order (ε^2) equations, we have

$$q_{y_i}^{(0)} = -k_i\frac{\partial T_i^{(1)}}{\partial y}. \tag{27}$$

Integration of equation (27) yields

$$T_i^{(1)} = -\frac{1}{k_i}\left(\frac{a_i}{2}y^2+b_i y+c_i\right) \tag{28}$$

where $c_i = c_i(x,t)$.

If we expand T_i up to ε^2 and q_{y_i} up to ε^0, we have

$$T_i = T_i^{(0)} - \frac{1}{k_i}\left(\frac{a_i}{2}y^2+b_i y+c_i\right)+O(\varepsilon^4) \tag{29}$$

$$q_{y_i} = a_i y + b_i + O(\varepsilon^2). \tag{30}$$

The average of $T_i(x,y,t)$ and $q_{x_i}(x,y,t)$ is defined as

$$\bar{T}_i(x,t) = \frac{\int_{y_{i-1}}^{y_i} T_i(x,y,t)\,\mathrm{d}y}{y_i - y_{i-1}} \tag{31}$$

$$\bar{q}_{x_i}(x,t) = -k_i\frac{\partial \bar{T}_i}{\partial x}. \tag{32}$$

Substituting equation (30) into equation (11) and then applying the average as that defined in equations (31) and (32) yields

$$(y_i - y_{i-1})\left[\rho_i c_{pi}\frac{\partial \bar{T}_i}{\partial t}+\frac{\partial \bar{q}_{x_i}}{\partial x}\right] = -a_i(y_i - y_{i-1}),\quad i=1,2,3. \tag{33}$$

Similarly the average of equation (29) gives

$$\bar{T}_i = T_i^{(0)} - \frac{\varepsilon^2}{k_i}\left(\frac{y_i^2+y_i y_{i-1}+y_{i-1}^2}{6}a_i + \frac{y_i+y_{i-1}}{2}b_i + c_i\right). \tag{34}$$

Substituting equation (34) into equation (29), gives

$$T_i = \bar{T}_i + \frac{\varepsilon^2}{k_i}\left(\frac{y_i^2+y_i y_{i-1}+y_{i-1}^2}{6}a_i + \frac{y_i+y_{i-1}}{2}b_i\right) - \frac{\varepsilon^2}{k}\left(\frac{a_i}{2}y^2+b_i y\right). \tag{35}$$

Substituting equation (35) into the interface condition, i.e. equations (15) and (16), gives

$$\bar{T}_1 + \frac{\varepsilon^2}{k_1}\left(\frac{y_1^2}{6}a_1+\frac{y_1}{2}b_1\right) - \frac{\varepsilon^2}{k_1}\left(\frac{a_1}{2}y_1^2+b_1 y_1\right) = \bar{T}_2 + \frac{\varepsilon^2}{k_2}\left(\frac{y_2^2+y_1 y_2+y_1^2}{6}a_2+\frac{y_2+y_1}{2}b_2\right) - \frac{\varepsilon^2}{k_2}\left(\frac{a_2}{2}y_1^2+b_2 y_1\right) \tag{36}$$

$$\bar{T}_2 + \frac{\varepsilon^2}{k_2}\left(\frac{y_2^2+y_1 y_2+y_2^2}{6}a_2+\frac{y_2+y_1}{2}b_2\right) - \frac{\varepsilon^2}{k_2}\left(\frac{a_2}{2}y_2^2+b_2 y_2\right) = \bar{T}_3 + \frac{\varepsilon^2}{k_3}\left(\frac{y_3^2+y_3 y_2+y_3^2}{6}a_3 + \frac{y_3+y_2}{2}b_3\right) - \frac{\varepsilon^2}{k_2}\left(\frac{a_3}{2}y_2^2+b_3 y_2\right) \tag{37}$$

$$a_1 y_1 + b_1 = a_2 y_2 + b_2 \tag{38}$$

$$a_2 y_2 + b_2 = a_3 y_3 + b_3. \tag{39}$$

Similarly from the external boundary conditions, i.e. equations (14) and (17), we have

$$b_1 = \frac{h_0}{\varepsilon^2}\left[T_\infty - \bar{T}_1 - \frac{\varepsilon^2}{k_1}\left(\frac{y_1^2}{6}a_1 + \frac{y_1}{2}b_1\right)\right] \quad (40)$$

$$a_3 y_3 + b_3 = \frac{h_3}{\varepsilon^2}\left[\bar{T}_3 + \frac{\varepsilon^2}{k_3}\left(\frac{y_3^2+y_2y_3+y_2^2}{6}a_3 + \frac{y_3+y_2}{2}b_3\right) - \frac{\varepsilon^2}{k_3}\left(\frac{a_3}{2}y_3^2 + b_3y_3\right) - T_\infty\right]. \quad (41)$$

Equations (36)–(41) can be put into the following form:

$$\underline{M} = \underline{\underline{P}}\,\underline{V}. \quad (42)$$

where

$$\underline{M} = [T_\infty - \bar{T}_1, \bar{T}_1 - \bar{T}_2, \bar{T}_2 - \bar{T}_3, \bar{T}_3 - T_\infty, 0, 0]^{\mathrm{T}}$$

$$\underline{V} = [a_1, b_1, a_2, b_2, a_3, b_3]^{\mathrm{T}}$$

$$P_{11} = \frac{\varepsilon^2 y_1^2}{6k_1}, \quad P_{12} = \frac{\varepsilon^2 y_1^2}{2k_1} + \frac{\varepsilon^2}{h_0}$$

$$P_{13} = P_{14} = P_{15} = P_{16} = 0, \quad P_{21} = \frac{\varepsilon^2 y_1^2}{3k_1}$$

$$P_{22} = \frac{\varepsilon^2 y_1}{2k_1}, \quad P_{23} = \frac{\varepsilon^2(y_2^2 + y_1y_2 - 2y_1^2)}{6k_2}$$

$$P_{24} = \frac{\varepsilon^2(y_2 - y_1)}{2k_2}, \quad P_{25} = P_{26} = 0, \quad P_{31} = P_{32} = 0$$

$$P_{33} = \frac{\varepsilon^2(2y_2^2 - y_1y_2 - y_1^2)}{6k_2}, \quad P_{34} = \frac{\varepsilon^2(y_2 - y_1)}{2k_2}$$

$$P_{35} = \frac{\varepsilon^2(y_3^2 + y_2y_3 - 2y_2^2)}{6k_3}, \quad P_{36} = \frac{\varepsilon^2(y_3 - y_2)}{2k_3}$$

$$P_{41} = P_{42} = P_{43} = P_{44} = 0$$

$$P_{45} = \frac{\varepsilon^2 y_3}{h_3} + \frac{\varepsilon^2(2y_3^2 - y_2y_3 - 2y_2^2)}{6k_3}$$

$$P_{46} = \frac{\varepsilon^2}{h_3} + \frac{\varepsilon^2(y_3 - y_2)}{2k_3}, \quad P_{51} = y_1$$

$$P_{52} = 1, \quad P_{53} = -y_1, \quad P_{54} = -1$$

$$P_{55} = P_{56} = 0, \quad P_{61} = P_{62} = 0, \quad P_{63} = y_2$$

$$P_{64} = 1, \quad P_{65} = -y_2, \quad P_{66} = -1.$$

From equation (41), we have $\underline{V} = \underline{\underline{P}}^{-1}\underline{M}$ and hence a_i can be expressed as $\bar{T}_i$ and T_∞. Substituting a_i into equation (33), we have

$$\underline{\underline{C}}\frac{\partial \underline{\bar{T}}}{\partial t} - \underline{\underline{E}}\frac{\partial^2 \underline{\bar{T}}}{\partial x^2} = \underline{\underline{D}}\,\underline{\bar{T}} + \underline{G}T_\infty \quad (43)$$

where $\underline{\bar{T}} = [\bar{T}_1, \bar{T}_2, \bar{T}_3]^{\mathrm{T}}$

$$\underline{\underline{C}} = \begin{bmatrix} y_1\rho_1c_{p1} & 0 & 0 \\ 0 & (y_2-y_1)\rho_2c_{p2} & 0 \\ 0 & 0 & (y_3-y_2)\rho_3c_{p3} \end{bmatrix}$$

$$\underline{\underline{E}} = \begin{bmatrix} y_1k_1 & 0 & 0 \\ 0 & (y_2-y_1)k_2 & 0 \\ 0 & 0 & (y_3-y_2)k_3 \end{bmatrix}$$

$$D_{11} = -y_1[P^{-1}_{12} - P^{-1}_{11}], \quad D_{12} = -y_1[P^{-1}_{13} - P^{-1}_{12}]$$

$$D_{13} = -y_1[P^{-1}_{14} - P^{-1}_{13}]$$

$$D_{21} = (y_1 - y_2)[P^{-1}_{32} - P^{-1}_{31}]$$

$$D_{22} = (y_1 - y_2)[P^{-1}_{33} - P^{-1}_{32}]$$

$$D_{23} = (y_1 - y_2)[P^{-1}_{34} - P^{-1}_{33}]$$

$$D_{31} = (y_2 - y_3)[P^{-1}_{52} - P^{-1}_{51}]$$

$$D_{32} = (y_2 - y_3)[P^{-1}_{53} - P^{-1}_{52}]$$

$$D_{33} = (y_2 - y_3)[P^{-1}_{54} - P^{-1}_{53}]$$

$$\underline{G} = [-y_1(P^{-1}_{11} - P^{-1}_{14}), (y_1 - y_2)(P^{-1}_{31} - P^{-1}_{34}), (y_2 - y_3)(P^{-1}_{51} - P^{-1}_{54})]^{\mathrm{T}}.$$

Though tedious, it can be shown straightforwardly that $\underline{\underline{D}}$ is a symmetric matrix. The corresponding initial and boundary conditions for equation (43) are

$$\left.\begin{aligned} \bar{T}_i(x,0) &= T_\infty \\ \bar{T}_i(0,t) &= T_0 \\ \left.\frac{\partial \bar{T}_i}{\partial x}\right|_{x=1} &= 0 \end{aligned}\right\} \quad i = 1, 2, 3.$$

Let $\theta_i = \bar{T}_i - T_0$, and substitute equation (43) and its boundary conditions into the following form:

$$\frac{\partial \underline{\theta}}{\partial t} = -L\underline{\theta} + \underline{f}$$

$$\underline{\theta}(x,0) = (T_\infty - T_0)\underline{I}$$

$$\underline{\theta}(0,t) = \underline{0}$$

$$\left.\frac{\partial \underline{\theta}}{\partial x}\right|_{x=1} = 0 \quad (44)$$

where

$$\underline{\theta} = [\theta_1, \theta_2, \theta_3]^{\mathrm{T}}, \underline{0} = [0,0,0]^{\mathrm{T}}, \underline{I} = [1,1,1]^{\mathrm{T}}$$

$$L = -\underline{\underline{C}}^{-1}\left[\underline{\underline{E}}\frac{\partial^2}{\partial x^2} + \underline{\underline{D}}\right]$$

$$\underline{f} = \underline{\underline{C}}^{-1}\underline{\underline{D}}\,\underline{I}T_0 + \underline{\underline{C}}^{-1}\underline{G}T_\infty.$$

A direct sum space is defined as

$$\mathbf{H} = \mathbf{H}_1 \oplus \mathbf{H}_2 \oplus \mathbf{H}_3$$

where $\mathbf{H}_i = \{L_2[0,1]; p_i\}$ consists of function $\phi_i(x)$ defined in [0, 1]. p_i is the weighting function, $p_1 = y_1\rho_1c_{p1}$, $p_2 = (y_2-y_1)\rho_2c_{p2}$, $p_3 = (y_3-y_2)\rho_3c_{p3}$, $\phi_i(x)$ satisfies the Lebesque integral

$$\int_0^1 \phi_i^2(x)p_i\,\mathrm{d}x < \infty.$$

An element in $\mathbf{H}$ is an ordered triple and can be viewed as a vector. For $\underline{\phi} \in \mathbf{H}$ and $\underline{\psi} \in \mathbf{H}$, the inner product of $\underline{\phi}$ and $\underline{\psi}$ is defined as

$$\langle \underline{\phi}, \underline{\psi} \rangle = \sum_{i=1}^{3} \rho_i c_{pi}(y_i - y_{i-1}) \int_0^1 \phi_i(x)\psi_i(x)\,\mathrm{d}x$$

with $y_0 = 0$. Then $\mathbf{H}$ is a Hilbert space. The domain of operator $\mathbf{L}$ is defined as

$$D(L) = \{\underline{\phi} \in \mathbf{H}; L\underline{\phi} \in \mathbf{H}; \underline{\phi}(0,t) = \underline{0}, \underline{\phi}'|_{x=1}) = \underline{0}\}.$$

Then $\mathbf{L} = \{L, D(L)\}$ is a self-adjoint operator in $\mathbf{H}$, i.e.

$$\langle \mathbf{L}\underline{\phi}, \underline{\psi} \rangle = \langle \underline{\phi}, \mathbf{L}\underline{\psi} \rangle \text{ for every } \underline{\phi}, \underline{\psi} \in \mathbf{H}.$$

Since $\mathbf{L}$ is a compact and self-adjoint operator in $\mathbf{H}$, the associate eigenvalue problem is

$$\mathbf{L}\underline{W} = \lambda \underline{W}. \tag{45}$$

Equation (45) is a set of three coupled Sturm–Liouville problems. The solution of equation (45) consists of three branches of eigenvalue $\lambda_1^i, \lambda_2^i, \lambda_3^i, \ldots$, with corresponding eigenfunctions $W_1^i, W_2^i, W_3^i, \ldots$, $i = 1, 2, 3$. The operator $\mathbf{L}$ is very similar to that in the work of Arce and Ramkrishna [18], $\mathbf{L}^{-1}$ is not compact and has two branches of eigenvalues that have non-zero accumulation points. According to the linear operator theory [19], these eigenvalues are all real and positive. Any vector $\underline{\theta} \in \mathbf{H}$ can be represented as

$$\underline{\theta} = \sum_{i=1}^{3} \sum_{j=1}^{\infty} \frac{\langle \underline{\theta}, \underline{W}_j^i \rangle}{\sqrt{\langle \underline{W}_j^i, \underline{W}_j^i \rangle}} \underline{W}_j^i \tag{46}$$

where $\langle \underline{\theta}, \underline{W}_j^i \rangle$ can be found from the solution of the following equation which is obtained by taking the inner product of equation (44) with $\underline{W}_j^i$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\langle \underline{\theta}, \underline{W}_j^i \rangle &= -\langle \mathbf{L}\underline{\theta}, \underline{W}_j^i \rangle + \langle \underline{f}, \underline{W}_j^i \rangle \\ &= -\lambda_j^i \langle \underline{\theta}, \underline{W}_j^i \rangle + \langle \underline{f}, \underline{W}_j^i \rangle. \end{aligned} \tag{47}$$

The solution of equation (47) with the initial condition $\langle \underline{\theta}, \underline{W}_j^i \rangle = \langle \underline{\theta}_0, \underline{W}_j^i \rangle$ at $t = 0$ is

$$\langle \underline{\theta}, W_j^i \rangle = \langle \underline{\theta}_0, W_j^i \rangle\, \mathrm{e}^{-\lambda_j^i t} + \mathrm{e}^{-\lambda_j^i t} \int_0^t \mathrm{e}^{-\lambda_j^i \tau} \langle \underline{f}, W_j^i \rangle\, \mathrm{d}\tau. \tag{48}$$

It is clear from the above description that after applying the continuum mixture theory and the linear operator method the solution of the problem of transient conduction in a rectangular trilaminated fin becomes straightforward if the solution of equation (45) is available. Equation (45) is a set of three coupled Sturm–Liouville problems. Though conceptually simple to solve, it is by no means simple even to resolve the numerical method [20]. Hsiao [21] tried to solve equation (45) by the shooting method and found that unless the coupling between equations is very weak and that the initial guesses are very close to true values, numerical integration will diverge and solutions are difficult to obtain especially for the branch of eigenvalues with the smallest absolute value. The detail for the solution of equation (45) is given in the Appendix.

After obtaining the temperature distribution of the fin, the dimensionless heat transfer rate can be calculated as

$$Q = -\sum_{i=1}^{3} K_i \left.\frac{\partial \bar{\theta}_i}{\partial x}\right|_{x=0} (y_i - y_{i-1}). \tag{49}$$

RESULTS AND DISCUSSION

As described before, Chu *et al.* [15] were the first to solve the problem of the transient conduction in a rectangular, symmetrical, trilaminated fin. The unsteady two-dimensional energy equation was solved by taking the Laplace transformation with respect to the time domain first and the resulting equation was then solved by the eigenfunction expansion method. After the solution in the transformed plane was obtained, the Fourier series technique was then used to obtain its inverse transformation. This same problem is solved by the method proposed in this work and for a particular set of parameters, results obtained by this work and those of Chu *et al.* are presented in Fig. 2 for comparison. It can be seen that there are considerable differences in the temperature distribution especially when time is small. Since our results exactly match those of Ghoshdastidar and Mukhopadhyay [17] obtained by using the numerical method, we believe that the results of Chu *et al.* are incorrect especially when time is small since the use of the Fourier series method in the numerical inversion of

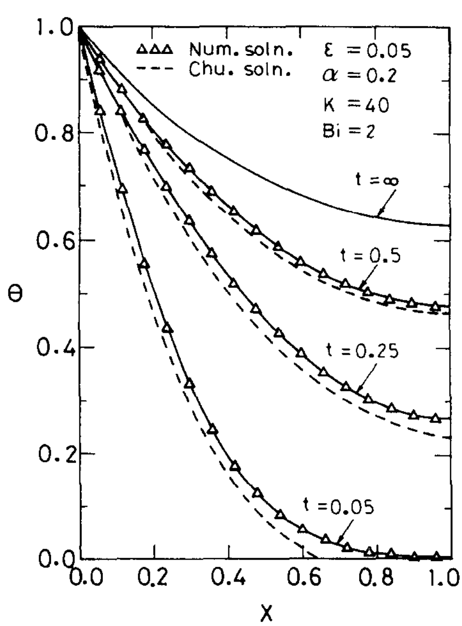

FIG. 2. Comparison of temperature distributions in the fin obtained by three different methods.

Table 1. Typical eigenvalues of a coupled Sturm–Liouville problem

No.	μ_j	λ_j^1	λ_j^2	λ_j^3
1	2.46740110	28 049.56628010	1502.19993963	17.71347997
2	22.20660990	28 071.82790413	1504.72373586	35.92378524
3	61.68502751	28 116.35166377	1509.88454167	72.23067084
4	120.90265391	28 183.13857639	1517.92783729	126.38763917
5	199.85948912	28 272.19015366	1529.27484499	197.97195630
6	298.55553313	28 383.50838456	1544.60313677	286.30406125
7	416.99078595	28 517.09571342	1564.98885346	390.30536884
8	555.16524756	28 672.95501330	1592.15358156	508.25141954
9	713.07891798	28 851.08955533	1628.88038883	637.35587445
10	890.73179720	29 051.50297460	1679.64828753	773.13608620

the Laplace transform causes the solution to converge too slow [16] to obtain a reasonably accurate solution.

Besides being simpler, the method proposed in this work can be used to deal with asymmetrical trilaminated fins. Asymmetry is caused either by a different coating on two sides of the base material or by a different convective heat transfer coefficient on two sides of the fin. Consider the problem of transient heat conduction in a three-layer asymmetrical rectangular fin. The base material is copper and has a dimensionless thickness of 0.9. On one side is a layer of stainless steel with a thickness of 0.08 and on the other side is a layer of gold with a thickness of 0.02. Convective heat transfer coefficients on both sides are assumed to be equal and $B_i = h_0/\varepsilon = h_3/\varepsilon = 2.0$, $\varepsilon = 0.1$. Table 1 shows three branches of the first ten eigenvalues (λ_j^i, $i = 1, 2, 3$; $j = 1, 2, \ldots, 10$) of equation (45) together with eigenvalues in the transformed domain (μ_j, $j = 1, 2, \ldots, 10$).

As discussed by Chu *et al.* [15], the thermal conductivity ratio is the most important parameter that affects the performance of the fin. Figure 3 shows the effect of the ratio of thermal conductivity of the base material to that of the coating material on the dimensionless rate of heat transfer for a three-layer symmetric fin. Since the ratio of thermal conductivity of copper to stainless steel is more than 20, it is clear from Fig. 3 that the symmetrical coating of stainless steel on copper seriously hampers the rate of heat transfer of the fin. The addition of stainless steel is sometimes necessary since it both increases the strength of the fin and at the same time protects the base material from the harsh environment. Are there other designs of a trilaminated fin that could add enough material strength and protect the base material without seriously hampering the heat transfer rate of the fin? The answer is the asymmetrical trilaminated fin described previously. With other parameters fixed, the performances of three different designs on a trilaminated fin are shown in Fig. 4 for comparison. Curve 1 is for a fin made of copper. Curve 2 is for an asymmetrical trilaminated fin made of three different materials; the base material is copper with a thickness of 0.9, on one side is a coating of stainless steel with a thickness of 0.09 and on the other side is a coating of gold with a thickness of 0.01. Curve 3 is for an asymmetrical trilaminated fin made of two different materials; the base material is copper with a

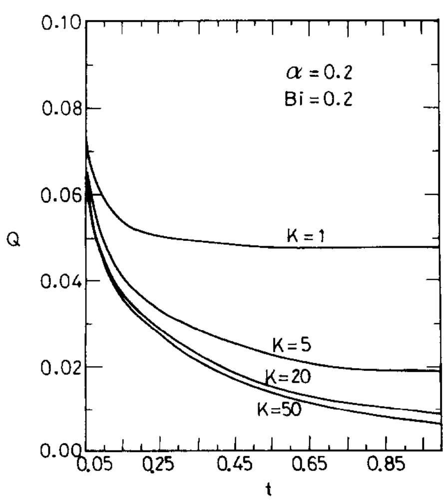

FIG. 3. Effects of thermal conductivity ratio on the performance of a symmetric trilaminated fin.

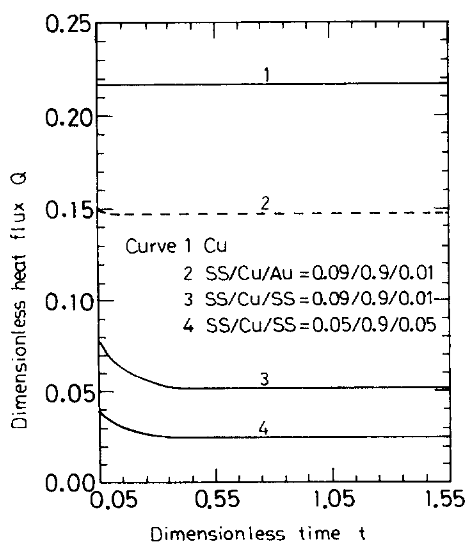

FIG. 4. Comparison of the performances of four kinds of design of rectangular fin.

thickness of 0.9 and on two sides of the base material there is a layer of stainless steel with thickness of 0.09 and 0.01, respectively. Curve 4 is for a symmetrical trilaminated fin; copper is the base material with a thickness of 0.9, there is a layer of stainless steel with a thickness of 0.05 on each side of the base material. From curves 3 and 4 of Fig. 4, it is clear that with other parameters fixed, an asymmetrical fin performs better than the symmetrical one. Since gold has a considerably higher thermal conductivity than stainless steel, the replacement of stainless steel with a thin layer of gold on one side of the base material greatly increases the efficiency of the trilaminated fin. Hence the best design for a trilaminated fin should be asymmetrical with copper as the base material and most stainless steel added on one side of the copper for the purpose of increasing the strength of the fin; the other side of the fin should be a thin layer of gold or if not economically possible, a layer of stainless steel as thin as possible.

CONCLUSION

We have successfully applied the continuum mixture theory and linear operator method in solving the problem of transient conduction in a rectangular, trilaminated fin. We have shown that an asymmetrical composite fin performs better than the symmetrical counterpart in terms of fin efficiency. The method proposed in this work can be applied to a composite fin of other geometry without difficulty.

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APPENDIX

Consider the following set of coupled Sturm–Liouville problems

$$[r(x)\underline{Y}']' + [q(x)\underline{\underline{I}} + p(x)\underline{\underline{A}}]\underline{Y} = 0 \tag{A1}$$

with boundary conditions

$$\underline{Y}(a) + \alpha\underline{\underline{I}}\underline{Y}'(a) = \underline{0} \tag{A2}$$

$$\underline{Y}(b) + \beta\underline{\underline{I}}\underline{Y}'(b) = \underline{0} \tag{A3}$$

where $r(x)$, $q(x)$ and $p(x)$ are piecewise continuous functions in $[a, b]$, $\underline{Y} = [y_1(x), y_2(x), \ldots, y_n(x)]^T$, $\underline{\underline{A}} = [A_{ij}]$ and $A_{ij} = a_{ij} + \delta_{ij}b_i\lambda$ (no summation). Assuming that μ is the eigenvalue of $\underline{\underline{A}}$ with the corresponding eigenvector $\underline{V}$ then we have

$$\underline{\underline{A}}\underline{V} = \mu\underline{V}$$

or

$$\det[\underline{\underline{A}} - \mu\underline{\underline{I}}] = 0. \tag{A4}$$

Equation (A4) is an nth order binomial in μ and λ, i.e.

$$a_0\lambda^n + a_1\lambda^{n-1}\mu + a_2\lambda^{n-2}\mu^2 + \cdots + a_{n-1}\lambda\mu^{n-1} + a_n\mu^n = 0. \tag{A5}$$

For a fixed λ, assume that equation (A5) has n distinct roots $\mu_1, \mu_2, \ldots, \mu_n$ with the corresponding eigenvectors $\underline{V}_1, \underline{V}_2, \ldots, \underline{V}_n$. Then from elementary matrix theory, $\underline{\underline{S}} = [\underline{V}_1, \underline{V}_2, \ldots, \underline{V}_n]$ is nonsingular and we have

$$\underline{\underline{S}}^{-1}\underline{\underline{A}}\underline{\underline{S}} = \underline{\underline{B}} \tag{A6}$$

where

$$\underline{\underline{B}} = [B_{ij}], B_{ij} = \delta_{ij}\mu_j \text{ (no summation).} \tag{A7}$$

Hence if we assume that $\underline{S}$ is known, let $\underline{Y} = \underline{S}\underline{\phi}$ and substituting it into equations (A1)–(A3), we have

$$[r(x)\underline{\phi}']' + [q(x)\underline{I} + P(x)\underline{B}]\underline{\phi} = 0 \tag{A8}$$

$$\underline{\phi}(a) + \alpha\underline{I}\underline{\phi}'(a) = 0 \tag{A9}$$

$$\underline{\phi}(b) + \beta\underline{I}\underline{\phi}'(b) = 0. \tag{A10}$$

Though equations (A8)–(A10) are still a set of Sturm–Liouville problems, however, they are decoupled. The three Sturm–Liouville problems are exactly the same and are of the following form:

$$[r(x)\phi']' + [q(x) + P(x)\mu]\phi = 0 \tag{A11}$$

$$\phi(a) + \alpha\phi'(a) = 0 \tag{A12}$$

$$\phi(b) + \beta\phi'(b) = 0. \tag{A13}$$

Eigenvalues μ_j with corresponding eigenfunctions ϕ_j of equations (A11)–(A13) can be solved easily by numerical integration [22]. Eigenvalues and eigenfunctions of equations (A1)–(A3) can then be obtained by substituting μ_j and ϕ_j into equation (A5) and $\underline{Y} = \underline{S}\underline{\phi}$.

CONCEPTION D'UNE AILETTE RECTANGULAIRE TRILAMINEE

Résumé—Une théorie de continuum combinée avec la méthode de l'opérateur linéaire est utilisée pour résoudre le problème de la conduction thermique variable dans une ailette rectangulaire trilaminée. En appliquant cette approche on réduit ce problème à un problème couplé de Sturm–Liouville. Une méthode est développée pour trouver les valeurs propres et les fonctions propres du problème. On illustre la conduction variable dans des ailettes symétriques et asymétriques. On trouve que l'ailette asymétrique est plus efficace que la disymétrique.

BERECHNUNG EINER DREILAGIGEN RECHTECKRIPPE

Zusammenfassung—Mit Hilfe einer Kontinuums-Misch-Theorie und eines linearen Operatorverfahrens wird das Problem der transienten Wärmeleitung in einer dreilagigen Rechteckrippe gelöst. Durch Anwendung dieses Verfahrens vereinfacht sich das Problem auf ein gekoppeltes Sturm–Liouville-Problem. Zur Bestimmung der Eigenwerte und der Eigenfunktionen des gekoppelten Sturm–Liouville-Problems wird ein besonderes Verfahren entwickelt. Die transiente Wärmeleitung in symmetrischen und in nicht-symmetrischen Rechteckrippen wird gezeigt. Es zeigt sich, daß im Hinblick auf den Wirkungsgrad sich die asymmetrische Rippe besser verhält als die symmetrische.

РАСЧЕТ ТРЕХСЛОЙНОГО ПРЯМОУГОЛЬНОГО РЕБРА

Аннотация—Теория сплошных сред в сочетании с линейным операторным методом используется для решения задачи нестационарной теплопроводности в прямоугольном трехслойном ребре. Их применение позволяет свести решаемую задачу к сопряженной задаче Штурма–Лиувилля. Разрабатывается метод отыскания собственных значений и собственных функций этой задачи. Рассчитывается нестационарная теплопроводность как в симметричных, так и в несимметричных прямоугольных ребрах. Найдено, что несимметричные ребра более Эффективны.